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Quantum 2 + 1 evolution model

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Abstract. A quantum evolution model in 2 + 1 discrete spacetime, connected with a 3D fundamental map $\mathbf{R}$, is investigated. Map $\mathbf{R}$ is derived as a map providing a zero curvature of a 2D *linear* lattice system called 'the current system'. In a special case of the local Weyl algebra for dynamical variables the map appears to be canonical and it corresponds to the known operator-valued $\mathbf{R}$-matrix. The current system is a type of the linear problem for the 2 + 1 evolution model. A generating function for the integrals of motion for the evolution is derived with the help of the current system. Thus, the complete integrability in 3D is proved directly.

Introduction

In 3D integrable models the tetrahedron equation (TE) takes the place of the Yang–Baxter equation (YBE) in two dimensions. Having obtained a solution of the TE, one may hope to construct a 3D integrable model. In the case of a finite number of states one may construct the usual layer-to-layer transfer matrices T, with which the TE commutes [1–3]. Such finite state models are usually interpreted as statistical mechanics models. In fact, only one such model still exists, the Zamolodchikov–Bazhanov–Baxter model [1, 3–5]. This uniqueness does not mean that the 3D world has no interest.

When 3D $\mathbf{R}$-matrices have infinitely many states, which is more usual in three dimensions, it is natural to investigate transfer matrices with no hidden space. We denote such transfer matrices as $\mathbf{U}$ as opposed to the notation for usual transfer matrix T. Matrices $\mathbf{U}$ commute with the set of T, but have no degrees of freedom when the set of T is fixed. Operator-valued matrices $\mathbf{U}$ are usually interpreted as evolution operators for systems associated with in-states of $\mathbf{U}$, making the map from in-states to out-states. In the realm of $1 + 1$ evolution models many such models associated with proper quantization of discrete equations have been derived, see [6, 7, 17] and references therein. Conventionally, models with infinitely many states are regarded as field theory models.

In three dimensions, $\mathbf{U}$-matrices, finite state as well as infinite state, geometrically appear as the element of a cubic lattice between two nearest inclined planes. We do not draw the graphical representation of a 3D $\mathbf{U}$ here; we consider sections of the cubic lattice made by the two, in- and out-, inclined planes mentioned. A 2D lattice appearing in such sections is called the Kagomé lattice and we consider it in detail below. Examples of U-matrices in three dimensions for finite state $\mathbf{R}$-matrix as well as examples of their eigenstates may be found in [8–10].

Here we derive our evolution model without considering any discrete 3D equation. Instead we derive a canonical map $\mathbf{R}$ as an intertwining operator between two algebraic objects, associated with the geometries of two Yang–Baxter graphs. In some sense our approach resembles the method of the local YBE proposed in [14–16] in which

$$L_{1,2}(x) \cdot L_{1,3}(y) \cdot L_{2,3}(x) = L_{2,3}(z') \cdot L_{1,3}(y') \cdot L_{1,2}(x'). \tag{0.1}$$

The intertwining operator between the left- and right-hand sides of the equation,

$$\mathsf{R} : [x, y, z] \mapsto [x', y', z'] \tag{0.2}$$

obeys the zero-curvature condition in 3D (the TE) automatically, because of the uniqueness of the solution

$$x' = x'(x, y, z) \qquad y' = y'(z, y, z) \qquad z' = z'(x, y, z) \tag{0.3}$$

of the local YBE. The key observation is that an intertwining functional operator, solving the TE, can be obtained from any other decent definition of an equivalence of two Yang–Baxter-type graphs [18, 28]. Here we formulate such a definition of the equivalence, that is an equivalence of some linear system [19, 20]. The linearity of the basic system allows us to derive a generating function for integrals of motion for the discrete evolution, governed by our intertwining map. The model we investigate in the present paper is the quantum counterpart of the functional evolution model considered briefly in [20].

This paper is organized as follows. In section 1 we formulate the linear system in general and describe the intertwining map. The map will become the unique and canonical one when we impose the local Weyl algebra conditions for the dynamical variables of the linear system. Being canonical, the map may be realized in terms of quantum dilogarithmic functions [23,24]. In section 2 we define the evolution. For that, the generating function for the integrals of motion is a properly defined determinant of the operator-valued matrix of the coefficients of the linear system. The determinant admits a combinatorial diagrammatic representation in terms of walks around the torus, on which the Kagomé lattice is defined. In these terms each integral of motion may be associated to a sum of the walks with a homotopy class fixed. As an example, we consider the simplest case of the evolution on a thin strip for a special limit of the intertwining operator. This corresponds to the quantum Liouville evolution. One can calculate the integrals of motion for this case explicitly. Finally, we discuss a host of unsolved problems and perspectives for further investigation. This paper is a journal version of the manuscript [30], where many associated questions are discussed in detail.

1. Auxiliary linear problem

In this section we give some rules allowing one to assign an algebraic system to a graph. The elements to which we assign something are vertices and sites. First, we give the most general rules, which do not give an algebraic equivalence of equivalent graphs in general, due to a kind of 'gauge ambiguity'. As a special case we find rules which do not contain a gauge ambiguity, and so a notion of algebraic equivalence can be introduced. Then we describe the map of the dynamical variables given by the equivalence of 2-simplices.

First of all, we fix some notation for the geometrical objects we deal with. Consider a graph G_n formed by n straight intersecting lines. The elements of its cw-complex are the vertices, the edges and the sites. G_n consists on $N_V = \frac{n(n-1)}{2}$ vertices, $N_S = \frac{(n-1)(n-2)}{2}$ closed inner sites and $N_S^* = 2n$ outer open sites, $N_E = n(n-2)$ closed inner edges and $N_E^* = 2n$ outer edges. If two graphs G_n and G'_n have the same outer structure, i.e. G'_n can be obtained from G_n by an appropriate shift of the lines, then we call G'_n and G_n equivalent. To such graphs we are going to associate an algebraic system, which gives an algebraic meaning to the geometrical equivalence, providing an intertwining map from parameters of G_n to parameters of G'_n.

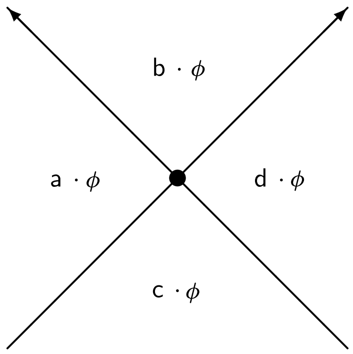

Figure 1. The current vertex.

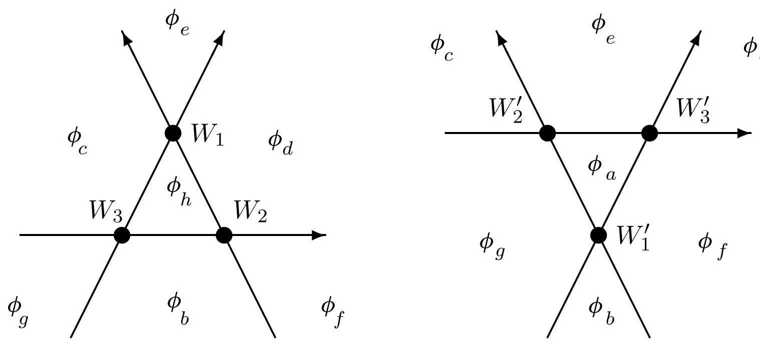

Figure 2. The Yang-Baxter equivalence $\triangle \sim \bigtriangledown$.

1.1. Linear system: general approach

Choose as a game the following rules:

- Assign to each oriented vertex V an auxiliary 'internal current' ϕ. Suppose this current produces four 'site currents' flowing from the vertex into four adjacent faces, and proportional to the internal current with some coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, called the dynamical variables, as is shown in figure 1. All these variables, ϕ and $\mathbf{a}, \ldots, \mathbf{d}$ for different vertices are independent for a while. We ask nothing of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, ϕ *a priori*, except for linearity with respect to ϕ and the right action of the coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ on ϕ.
- Define the complete site current as an algebraic sum of the contributions of vertices surrounding this site.
- For any closed site of a lattice let its complete current be zero. Such zero relations we regard as the linear equations for the internal currents.
- For any graph G_n the site currents assigned to outer (open) sites we call the 'observable currents'. Two equivalent graphs G_n and G_n' must have the same observable currents—this is the algebraic meaning of equivalence.

We clarify these rules with the example of the equivalence of G_3. As was mentioned, this is the usual Yang–Baxter equivalence graphically, schematically $\triangle = \bigtriangledown$, shown in figure 2. Here, for brevity, we denote the left-hand configuration of G_3 as $\triangle$, and the right-hand configuration as $\bigtriangledown$. We assign to the vertices W_j of $\triangle$ the currents ϕ_j and dynamical variables $\mathbf{a}_j$, $\mathbf{b}_j$, $\mathbf{c}_j$, $\mathbf{d}_j$ and to the vertices W_j' of $\bigtriangledown$, the currents ϕ_j' and dynamical variables $\mathbf{a}_j'$, $\mathbf{b}_j'$, $\mathbf{c}_j'$, $\mathbf{d}_j'$. Six currents of outer sites are denoted as $\phi_b, \ldots, \phi_g$, and two zero-valued currents of closed sites as ϕ_h and ϕ_a as shown in figure 2. Then, using the rules described above, we obtain the following system of eight linear (with respect to the currents) relations:

$$\phi_h \equiv \mathbf{c}_1 \cdot \phi_1 + \mathbf{a}_2 \cdot \phi_2 + \mathbf{b}_3 \cdot \phi_3 = 0 \tag{1.1}$$

$$\begin{aligned} \phi_b &\equiv \mathbf{c}'_1 \cdot \phi'_1 = \mathbf{c}_2 \cdot \phi_2 + \mathbf{d}_3 \cdot \phi_3 \\ \phi_c &\equiv \mathbf{a}'_2 \cdot \phi'_2 = \mathbf{a}_1 \cdot \phi_1 + \mathbf{a}_3 \cdot \phi_3 \\ \phi_d &\equiv \mathbf{b}'_3 \cdot \phi'_3 = \mathbf{d}_1 \cdot \phi_1 + \mathbf{b}_2 \cdot \phi_2 \end{aligned} \tag{1.2}$$

$$\begin{aligned} \phi_e &\equiv \mathbf{b}'_2 \cdot \phi'_2 + \mathbf{a}'_3 \cdot \phi'_3 = \mathbf{b}_1 \cdot \phi_1 \\ \phi_f &\equiv \mathbf{d}'_1 \cdot \phi'_1 + \mathbf{d}'_3 \cdot \phi'_3 = \mathbf{d}_2 \cdot \phi_2 \\ \phi_g &\equiv \mathbf{a}'_1 \cdot \phi'_1 + \mathbf{c}'_2 \cdot \phi'_2 = \mathbf{c}_3 \cdot \phi_3 \end{aligned} \tag{1.3}$$

$$\phi_a \equiv \mathbf{b}'_1 \cdot \phi'_1 + \mathbf{d}'_2 \cdot \phi'_2 + \mathbf{c}'_3 \cdot \phi'_3 = 0. \tag{1.4}$$

These give the currents and the dynamical variables for $\triangle$. As $\phi_h = 0$, equation (1.1), only two currents are independent; let them be ϕ_1 and ϕ_3. All the variables for $\triangledown$ we try to restore via the linear system: first, use ϕ_b, ϕ_c and ϕ_d (1.2) to express all ϕ'_j; substitute ϕ'_j into relations for ϕ_e, ϕ_f and ϕ_g (1.3), then three homogeneous linear relations for two arbitrary ϕ_1 and ϕ_3 will appear, so six coefficients of ϕ_1 and ϕ_3 must vanish. Solving these six equations with respect to the primed variables, we obtain

$$\begin{aligned} &\mathbf{b}'_2\mathbf{a}'^{-1}_2 = \Lambda_1^{-1} \cdot \mathbf{b}_3\mathbf{a}_3^{-1} \qquad && \mathbf{a}'_3\mathbf{b}'^{-1}_3 = \Lambda_1^{-1} \cdot \mathbf{a}_2\mathbf{b}_2^{-1} \\ &\mathbf{d}'_1\mathbf{c}'^{-1}_1 = \Lambda_2^{-1} \cdot \mathbf{b}_3\mathbf{d}_3^{-1} && \mathbf{d}'_3\mathbf{b}'^{-1}_3 = \Lambda_2^{-1} \cdot \mathbf{c}_1\mathbf{d}_1^{-1} \\ &\mathbf{a}'_1\mathbf{c}'^{-1}_1 = \Lambda_3^{-1} \cdot \mathbf{a}_2\mathbf{c}_2^{-1} && \mathbf{c}'_2\mathbf{a}'^{-1}_2 = \Lambda_3^{-1} \cdot \mathbf{c}_1\mathbf{a}_1^{-1} \end{aligned} \tag{1.5}$$

where three polynomials have arisen:

$$\begin{aligned} \Lambda_1 &= \mathbf{b}_3\mathbf{a}_3^{-1}\mathbf{a}_1\mathbf{b}_1^{-1} - \mathbf{c}_1\mathbf{b}_1^{-1} + \mathbf{a}_2\mathbf{b}_2^{-1}\mathbf{d}_1\mathbf{b}_1^{-1} \\ \Lambda_2 &= \mathbf{b}_3\mathbf{d}_3^{-1}\mathbf{c}_2\mathbf{d}_2^{-1} - \mathbf{a}_2\mathbf{d}_2^{-1} + \mathbf{c}_1\mathbf{d}_1^{-1}\mathbf{b}_2\mathbf{d}_2^{-1} \\ \Lambda_3 &= \mathbf{a}_2\mathbf{c}_2^{-1}\mathbf{d}_3\mathbf{c}_3^{-1} - \mathbf{b}_3\mathbf{c}_3^{-1} + \mathbf{c}_1\mathbf{a}_1^{-1}\mathbf{a}_3\mathbf{c}_3^{-1}. \end{aligned} \tag{1.6}$$

Substituting ϕ'_j into $\phi_a = 0$ (1.4), we obtain the homogeneous linear equation for ϕ_1, ϕ_3 again, and the coefficients of them vanish if

$$\begin{aligned} \mathbf{b}'_1\mathbf{c}'^{-1}_1 &= \Lambda_a\Lambda_1(\mathbf{c}_2\mathbf{b}_2^{-1}\mathbf{d}_1\mathbf{b}_1^{-1} + \mathbf{d}_3\mathbf{a}_3^{-1}\mathbf{a}_1\mathbf{b}_1^{-1})^{-1} \\ \mathbf{d}'_2\mathbf{a}'^{-1}_2 &= \Lambda_a\Lambda_2(\mathbf{a}_1\mathbf{d}_1^{-1}\mathbf{b}_2\mathbf{d}_2^{-1} + \mathbf{a}_3\mathbf{d}_3^{-1}\mathbf{c}_2\mathbf{d}_2^{-1})^{-1} \\ \mathbf{c}'_3\mathbf{b}'^{-1}_3 &= \Lambda_a\Lambda_3(\mathbf{d}_1\mathbf{a}_1^{-1}\mathbf{a}_3\mathbf{c}_3^{-1} + \mathbf{b}_2\mathbf{c}_2^{-1}\mathbf{d}_3\mathbf{c}_3^{-1})^{-1} \end{aligned} \tag{1.7}$$

where Λ_a is arbitrary. The origin of Λ_a technically is $\phi_a = \Lambda_a \cdot \phi_h$.

This Λ_a is a kind of gauge. The origin of it is that as $\phi_a \equiv 0$ we may change $\phi_a \mapsto \lambda_a\phi_a$; this gives $\Lambda_a \mapsto \lambda_a\Lambda_a$, or equivalently

$$\mathbf{b}'_1 \mapsto \lambda_a\mathbf{b}'_1 \qquad \mathbf{d}'_2 \mapsto \lambda_a\mathbf{d}'_2 \qquad \mathbf{c}'_3 \mapsto \lambda_a\mathbf{c}'_3. \tag{1.8}$$

The analogous degree of freedom is lost in the map $W_1, W_2, W_3 \mapsto W'_1, W'_2, W'_3$: the system of the observables is not changed when $\phi_h \mapsto \lambda_h\phi_h$, i.e. when

$$\mathbf{c}_1 \mapsto \lambda_h\mathbf{c}_1 \qquad \mathbf{a}_2 \mapsto \lambda_h\mathbf{a}_2 \qquad \mathbf{b}_3 \mapsto \lambda_h\mathbf{b}_3 \tag{1.9}$$

and the formulae for W'_j do not change with (1.9). We call such invariance of the system of the observables *the site projective invariance* (correspondingly, the site ambiguity of the dynamical variables).

The other obvious invariance (ambiguity) is *the vertex projective* one. As a consequence of simple re-scaling of the currents almost nothing changes if

$$\mathbf{a} \mapsto \mathbf{a}\lambda \qquad \mathbf{b} \mapsto \mathbf{b}\lambda \qquad \mathbf{c} \mapsto \mathbf{c}\lambda \qquad \mathbf{d} \mapsto \mathbf{d}\lambda \tag{1.10}$$

partially in all vertices W_j and W'_j with six different λ_j and λ'_j.

Thus, in the most general interpretation, the map $W_1, W_2, W_3 \mapsto W'_1, W'_2, W'_3$ is defined up to projective ambiguity $\lambda_1, \lambda_2, \lambda_3, \lambda_h \mapsto \lambda'_1, \lambda'_2, \lambda'_3, \lambda_a$.

A very important feature of all these calculations is that *we never tried to commute anything*!

We return to a general case of graph G_n. $4N_V = 2n(n-1)$ free invertible variables $\mathbf{a}_V, \mathbf{b}_V, \mathbf{c}_V, \mathbf{d}_V$, assigned to the vertices V of G_n, we regard as the generators of a body $\mathcal{B}(G_n)$. For an open graph G_n one may consider $\mathcal{B}'(G_n)$, the set of functions invariant with respect to both vertex and closed site ambiguities. It is easy to see that the basis of $\mathcal{B}'(G_n)$ is formed by $4N_V - N_V - N_S = n^2 - 1$ independent monomials.

Consider a little change of G_n, so that only one $\triangle$ in G_n transforms into $\bigtriangledown$. Call the resulting graph G'_n. Let the vertices involved in this change be marked as W_1, W_2, W_3 for $\triangle$ and W'_1, W'_2, W'_3 for $\bigtriangledown$, arranged as in figure 2. We introduce a functional operator $\mathbf{R} = \mathbf{R}_{1,2,3}$ making the corresponding intertwining map on $\mathcal{B}$:

$$\mathbf{R}_{1,2,3} \cdot \varphi(W_1, W_2, W_3, \ldots) \cdot \mathbf{R}_{1,2,3}^{-1} = \varphi(W'_1, W'_2, W'_3, \ldots) \qquad \varphi \in \mathcal{B} \tag{1.11}$$

where W_j stands for $\{\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j, \mathbf{d}_j\}$ forever, and all other vertices except W_1, W_2, W_3 and their variables remain untouched. This ambiguous $\mathbf{R}$ we call the *fundamental map*.

Now, let G'_n be an arbitrary graph equivalent to G_n. G'_n can be obtained from G_n by different sequences of elementary $\triangle \mapsto \bigtriangledown$ in general. Thus the corresponding different sequences of $\mathbf{R}$ must coincide; this is the natural zero-curvature condition for $G_n \mapsto G'_n$. Partially, for the equivalence of G_4, the corresponding relation is the TE

$$\mathbf{R}_{1,2,3} \cdot \mathbf{R}_{1,4,5} \cdot \mathbf{R}_{2,4,6} \cdot \mathbf{R}_{3,5,6} = \mathbf{R}_{3,5,6} \cdot \mathbf{R}_{2,4,6} \cdot \mathbf{R}_{1,4,5} \cdot \mathbf{R}_{1,2,3}. \tag{1.12}$$

Due to the ambiguity of $\mathbf{R}$, (1.8), (1.9), any zero-curvature condition is still an equation for a set of Λ_ath involved. Recall, $\mathcal{B}'(G_n)$ was introduced previously as the gauge-invariant subspace of $\mathcal{B}$. $\mathbf{R}$ acts on $\mathcal{B}'$ uniquely. The number of linearly independent currents of G_n is $N_v - N_S = n - 1$, so the linear system actually has $n+1$ bounds for $N_S^* = 2n$ outer currents. This corresponds to $(n-1)(n+1) = n^2 - 1$ independent coefficients of the whole linear system, i.e. the principal number of equations coincides with the dimension of the basis of $\mathcal{B}'$. Unfortunately, the basis of $\mathcal{B}'$ is not local, and it is simpler to introduce an algebra constraint removing the projective ambiguities than to consider $\mathcal{B}'$ formally.

A way to remove Λ_a ambiguity from the definition of $\mathbf{R}$*, (1.5), (1.7), is to impose some additional conditions for the elements of W,* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$*, such that (1.7) would become a consequence of (1.5) and the additional conditions.*

Complete classification of these additional conditions is still an open problem.

1.2. Local case: the Weyl algebra

Here we consider a special *local* case: suppose first that the elements of two different W_i and W_j for given G_n commute. Destroy also the vertex projective invariance choosing $\mathbf{a}_j \equiv 1$ for any j forever. Then (1.5) give the expressions for $\mathbf{b}'_2$, $\mathbf{b}'_3$, $\mathbf{d}'_1\mathbf{c}'^{-1}_1$, $\mathbf{d}'_3\mathbf{b}'^{-1}_3$, $\mathbf{c}'_1$, $\mathbf{c}'_2$. Suppose also that any pair of the variables from W are linearly independent, then

- the commutativity of the elements for different W'_j from $\bigtriangledown$ gives (after some calculations) $\mathbf{bc} = q\mathbf{cb}$ with the same $\mathcal{C}$-number q for any vertex;

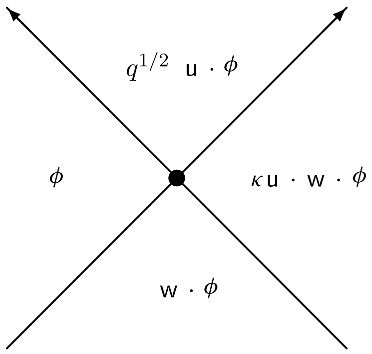

Figure 3. Local parametrization of the vertex.

- these relations are conserved by the map $\mathbf{R}$, i.e. $\mathbf{b}'\mathbf{c}' = q\mathbf{c}'\mathbf{b}'$;
- $\mathbf{b}^{-1}\mathbf{c}^{-1}\mathbf{d}$ also appear to be centres, depending on the vertex.

The gauge ambiguity becomes the ambiguity for these centres. We are looking for a kind of quantum theory, $\mathbf{b}$ and $\mathbf{c}$ are already quantized, so we have to keep all centres invariant, $\mathbf{b}_j^{-1}\mathbf{c}_j^{-1}\mathbf{d}_j = \mathbf{b}_j'^{-1}\mathbf{c}_j'^{-1}\mathbf{d}_j'$. This is possible, and further we will treat these centres as a kind of spectral parameter.

We now change notations for the dynamical variables to more conventional ones, and write down the resulting expressions for the map $\mathbf{R}$. New notations for the site currents are shown in figure 3.

Proposition 1. *Let the vertex dynamical variables be given by*

$$\mathbf{a} = 1 \qquad \mathbf{b} = q^{1/2}\mathbf{u} \qquad \mathbf{c} = \mathbf{w} \qquad \mathbf{d} = \kappa\mathbf{u}\mathbf{w}. \tag{1.13}$$

Here $\mathbf{u}$, $\mathbf{w}$ *obey the local Weyl algebra relation,*

$$\mathbf{u}\cdot\mathbf{w} = q\mathbf{w}\cdot\mathbf{u}. \tag{1.14}$$

$\mathbf{u}$ *and* $\mathbf{w}$ *for different vertices commute, and number* κ *is the invariant of the vertex, i.e.* $\kappa_{i,j}$, *assigned to the intersection of lines* i *and* j, *is the same for all equivalent graphs.*

Then the problem of the algebraic equivalence (i.e. equality of the outer currents) of two graphs: G *with the data* $\phi, \mathbf{u}, \mathbf{w}$, *and* G' *with the data* $\phi', \mathbf{u}', \mathbf{w}'$, *can be solved* without any ambiguity *with respect to all* $\phi', \mathbf{u}', \mathbf{w}'$, *and the local Weyl algebra structure for the set of* $\mathbf{u}', \mathbf{w}'$ *is the consequence of the local Weyl algebra relations for the set of* $\mathbf{u}, \mathbf{w}$.

We write the fundamental simplex map for $\triangle = \triangledown$ explicitly. The map $\mathbf{R} = \mathbf{R}_{1,2,3} : W_1, W_2, W_3 \mapsto W_1', W_2', W_3'$,

$$\mathbf{R}\cdot\mathbf{u}_j = \mathbf{u}_j'\cdot\mathbf{R} \qquad \mathbf{R}\cdot\mathbf{w}_j = \mathbf{w}_j'\cdot\mathbf{R} \qquad j = 1, 2, 3 \tag{1.15}$$

is given by

$$\begin{aligned} &\mathbf{w}_1' = \mathbf{w}_2\cdot\Lambda_3 \qquad \mathbf{u}_1' = \Lambda_2^{-1}\cdot\mathbf{w}_3^{-1} \\ &\mathbf{w}_2' = \Lambda_3^{-1}\cdot\mathbf{w}_1 \qquad \mathbf{u}_2' = \Lambda_1^{-1}\cdot\mathbf{u}_3 \\ &\mathbf{w}_3' = \Lambda_2^{-1}\cdot\mathbf{u}_1^{-1} \qquad \mathbf{u}_3' = \mathbf{u}_2\cdot\Lambda_1 \end{aligned} \tag{1.16}$$

where

$$\begin{aligned} \Lambda_1 &= \mathbf{u}_1^{-1}\cdot\mathbf{u}_3 - q^{1/2}\mathbf{u}_1^{-1}\cdot\mathbf{w}_1 + \kappa_1\mathbf{w}_1\cdot\mathbf{u}_2^{-1} \\ \Lambda_2 &= \frac{\kappa_1}{\kappa_2}\mathbf{u}_2^{-1}\cdot\mathbf{w}_3^{-1} + \frac{\kappa_3}{\kappa_2}\mathbf{u}_1^{-1}\cdot\mathbf{w}_2^{-1} - q^{-1/2}\frac{\kappa_1\kappa_3}{\kappa_2}\mathbf{u}_2^{-1}\cdot\mathbf{w}_2^{-1} \\ \Lambda_3 &= \mathbf{w}_1\cdot\mathbf{w}_3^{-1} - q^{1/2}\mathbf{u}_3\cdot\mathbf{w}_3^{-1} + \kappa_3\mathbf{w}_2^{-1}\cdot\mathbf{u}_3. \end{aligned} \tag{1.17}$$

Reverse formulae, giving $\mathbf{R}^{-1}$, look similar:

$$\begin{aligned}
\Lambda_1^{-1} &= \frac{\kappa_1}{\kappa_2}\mathbf{u}_1' \cdot \mathbf{u}_3'^{-1} - q^{1/2}\frac{\kappa_3}{\kappa_2}\mathbf{u}_1' \cdot \mathbf{w}_1'^{-1} + \kappa_3 \mathbf{w}_1'^{-1} \cdot \mathbf{u}_2' \\
\Lambda_2^{-1} &= \mathbf{u}_2' \cdot \mathbf{w}_3' + \mathbf{u}_1' \cdot \mathbf{w}_2' - q^{-1/2}\kappa_2 \mathbf{u}_2' \cdot \mathbf{w}_2' \\
\Lambda_3^{-1} &= \frac{\kappa_3}{\kappa_2}\mathbf{w}_1'^{-1} \cdot \mathbf{w}_3' - q^{1/2}\frac{\kappa_1}{\kappa_2}\mathbf{u}_3'^{-1} \cdot \mathbf{w}_3' + \kappa_1 \mathbf{w}_2' \cdot \mathbf{u}_3'^{-1}.
\end{aligned} \tag{1.18}$$

The conservation of the Weyl algebra structure

$$\mathbf{u}_j \cdot \mathbf{w}_j = q\mathbf{w}_j \cdot \mathbf{u}_j \mapsto \mathbf{u}_j' \cdot \mathbf{w}_j' = q\mathbf{w}_j' \cdot \mathbf{u}_j' \tag{1.19}$$

means that $\mathbf{R}$ is the canonical map, hence $\mathbf{R}_{1,2,3}$ can be regarded as usual operator depending on $\mathbf{u}_1, \mathbf{w}_1, \mathbf{u}_2, \mathbf{w}_2, \mathbf{u}_3, \mathbf{w}_3$. The structure of $\mathbf{R}$ will be described in section 1.3.

Now the projective ambiguity is removed, and the current system game gives unique correspondence between the elements of equivalent graphs. This is the exact meaning of algebraic equivalence. Hence all the zero-curvature conditions (and surely the tetrahedron relation) become trivial consequences of this unambiguity, and we get them gratis!

We mention now a couple of useful limits of our fundamental map $\mathbf{R}_{1,2,3}$. The first one is the limit when $\kappa_1 = \kappa_2 = \kappa_3 = \kappa$, and then $\kappa \mapsto 0$. We denote such limiting procedure via

$$\kappa_1 = \kappa_2 = \kappa_3 \ll 1. \tag{1.20}$$

The corresponding map we denote $\mathbf{R}_{1,2,3}^{pl}$. The conditions for κ are uniform for the whole tetrahedron relation,

$$\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa_5 = \kappa_6 \ll 1 \tag{1.21}$$

so $\mathbf{R}^{pl}$ obeys the TE. The other case is the limit of $\mathbf{R}_{1,2,3}$ when

$$\kappa_1 \ll \kappa_2 = \kappa_3 \ll 1. \tag{1.22}$$

These conditions are uniform for TE again,

$$\kappa_1 \ll \kappa_2 = \kappa_3 \ll \kappa_4 = \kappa_5 = \kappa_6 \ll 1. \tag{1.23}$$

We call the corresponding map $\mathbf{r}_{1,2,3}$ and due to the uniformnity it also obeys the TE. Recall, all these maps, $\mathbf{R}$ with $\kappa_1 = \kappa_2 = \kappa_3 = 1$, $\mathbf{R}^{pl}$ and $\mathbf{r}$, were derived previously as a hierarchy of $\mathbf{R}$-operators solving the TE, see [21, 22, 26, 29].

1.3. Structure of $\mathbf{R}$

We now give a realization of $\mathbf{R}$ in terms of simpler functions. First, recall the definition and properties of the quantum dilogarithm. Let, conventionally,

$$(\mathbf{x}; q)_n = (1 - \mathbf{x})(1 - q\mathbf{x})(1 - q^2\mathbf{x}) \ldots (1 - q^{n-1}\mathbf{x}). \tag{1.24}$$

Then the quantum dilogarithm (by definition) [23, 24]

$$\psi(\mathbf{x}) \stackrel{def}{=} (q^{1/2}\mathbf{x}; q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q; q)_n}\mathbf{x}^n \tag{1.25}$$

and

$$\psi(\mathbf{x})^{-1} = \sum_{n=0}^{\infty} \frac{q^{n/2}}{(q; q)_n} x^n. \tag{1.26}$$

This function is useful for the rational transformations of the Weyl algebra:

$$\psi(q\mathbf{x}) = (1 - q^{1/2}\mathbf{x})^{-1}\psi(\mathbf{x}) \qquad \psi(q^{-1}\mathbf{x}) = (1 - q^{-1/2}\mathbf{x})\psi(\mathbf{x}) \tag{1.27}$$

hence

$$\psi(\mathbf{u})\cdot\mathbf{w} = \mathbf{w}\cdot(1-q^{1/2}\mathbf{u})^{-1}\cdot\psi(\mathbf{u}) \qquad \psi(\mathbf{w})\cdot\mathbf{u} = \mathbf{u}\cdot(1-q^{-1/2}\mathbf{w})\cdot\psi(\mathbf{w}). \tag{1.28}$$

ψ is called the quantum dilogarithm due to the pentagon identity [23]

$$\psi(\mathbf{w})\cdot\psi(\mathbf{u}) = \psi(\mathbf{u})\cdot\psi(-q^{-1/2}\mathbf{u}\mathbf{w})\cdot\psi(\mathbf{w}). \tag{1.29}$$

This corresponds to Roger's five-term relation for the usual dilogarithm. From the other side ψ is the quantum exponent due to

$$\psi(\mathbf{u})\cdot\psi(\mathbf{w}) = \psi(\mathbf{u}+\mathbf{w}). \tag{1.30}$$

Recall, everywhere the Weyl algebra relation $\mathbf{u}\mathbf{w} = q\mathbf{w}\mathbf{u}$ is implied.

We now introduce a generalized permutation function. Let $\mathbf{P}(\mathbf{x},\mathbf{y})$, $\mathbf{x}\cdot\mathbf{y} = q^2\mathbf{y}\cdot\mathbf{x}$, defined by the following relations:

$$\begin{aligned}&\mathbf{P}(q\mathbf{x},\mathbf{y}) = \mathbf{y}^{-1}\mathbf{P}(\mathbf{x},\mathbf{y}) = \mathbf{P}(\mathbf{x},\mathbf{y})\mathbf{y}\\ &\mathbf{P}(\mathbf{x},q\mathbf{y}) = \mathbf{P}(\mathbf{x},\mathbf{y})\mathbf{x}^{-1} = \mathbf{x}\mathbf{P}(\mathbf{x},\mathbf{y})\end{aligned} \tag{1.31}$$

and

$$\mathbf{P}(\mathbf{x},\mathbf{y})^2 = 1. \tag{1.32}$$

For $\mathbf{z}$ obeying

$$\mathbf{x}\cdot\mathbf{z} = q^{f_x}\mathbf{z}\cdot\mathbf{x} \qquad \mathbf{y}\cdot\mathbf{z} = q^{f_y}\mathbf{z}\cdot\mathbf{y} \tag{1.33}$$

it follows that

$$\mathbf{P}(\mathbf{x},\mathbf{y})\cdot\mathbf{z} = q^{f_x f_y}\mathbf{z}\cdot\mathbf{x}^{f_y}\cdot\mathbf{y}^{-f_x}\cdot\mathbf{P}(\mathbf{x},\mathbf{y}). \tag{1.34}$$

This function we call the generalized permutation because the usual permutation operator of the tensor product is

$$\mathbf{P} \equiv \mathbf{P}(\mathbf{u}\otimes\mathbf{u}^{-1},\mathbf{w}\otimes\mathbf{w}^{-1}). \tag{1.35}$$

Considering independent $\mathbf{u}'_j\cdot\mathbf{u}_j^{-1}$ and $\mathbf{w}'_j\cdot\mathbf{w}_j^{-1}$, $j = 1,2,3$, one may see that they all depend on three operators $\mathbf{U}$, $\mathbf{W}$ and $\mathbf{s}$:

$$\mathbf{U} = \mathbf{w}_2^{-1}\cdot\mathbf{w}_3 \qquad \mathbf{W} = \mathbf{w}_1\cdot\mathbf{u}_3^{-1} \qquad -q^{1/2}\mathbf{s}\cdot\mathbf{U}\cdot\mathbf{W}^{-1} = \mathbf{u}_1\cdot\mathbf{u}_2^{-1}. \tag{1.36}$$

$\mathbf{U}\mathbf{W} = q\mathbf{W}\mathbf{U}$ and $\mathbf{s}$ is the centre. One can directly verify that

$$\mathbf{R} = \psi(\kappa_3\mathbf{U})\cdot\psi(\mathbf{W}^{-1})\cdot\mathbf{P}\left(\sqrt{\frac{\kappa_3}{\kappa_2}}\mathbf{U},\mathbf{s}^{-1}\cdot\mathbf{W}^2\right)\cdot\psi\left(\frac{\kappa_1}{\kappa_3}\mathbf{W}\right)^{-1}\cdot\psi(\kappa_2\mathbf{U}^{-1})^{-1} \tag{1.37}$$

being substituted into (1.15), gives (1.16), (1.17). On $\mathbf{U}$ and $\mathbf{W}$, $\mathbf{R}$ acts as follows:

$$\begin{aligned}&\mathbf{R}\cdot\mathbf{U}\cdot\mathbf{R}^{-1} = \frac{\kappa_2}{\kappa_3}\mathbf{U}^{-1}\cdot(\mathbf{W}-q^{-1/2}+\kappa_3\mathbf{U})\cdot\left(\mathbf{W}-q^{-1/2}\frac{\kappa_1}{\kappa_3}\mathbf{s}+\kappa_1\mathbf{s}\cdot\mathbf{U}\right)^{-1}\\ &\mathbf{R}\cdot\mathbf{W}\cdot\mathbf{R}^{-1} = \mathbf{s}\cdot\mathbf{W}^{-1}\cdot(\mathbf{W}-q^{1/2}+\kappa_3\mathbf{U})\cdot(\mathbf{W}-q^{1/2}+\kappa_1\mathbf{s}\cdot\mathbf{U})^{-1}.\end{aligned} \tag{1.38}$$

When $\kappa_1 = \kappa_2 = \kappa_3 = 1$, expression (1.37) for $\mathbf{R}$ coincides with the operator solution of the TE from [21, 22]. This is the generalization of the finite-dimensional 3D R-matrix from $q^N = 1$ to general q, and the finite-dimensional R-matrix corresponds to the Zamolodchikov–Bazhanov–Baxter model.

We do not discuss this correspondence here: the reader may find the details concerning the Zamolodchikov–Bazhanov–Baxter model in [1, 3, 4, 12], the details concerning the finite R-matrix in [5], the details concerning the quantum dilogarithm in the original papers [23, 24], and operator-valued $\mathbf{R}$ as the generalization of finite R in [21, 22, 26, 27].

We now consider the significance of (1.37). All ψ can be decomposed into the series with respect to their arguments. Substitute these $\mathbf{R}$ into the tetrahedron relation (1.12) and move all the generalized permutations $\mathbf{P}$ out. $\mathbf{P}$ themselves obey the TE and so can be cancelled from the TE for $\mathbf{R}$. Then 12 ψ are in the left-hand side of the TE, and 12 in the right-hand side. In this case the TE becomes a relation resembling the braid group relation in 2D. This 24 term relation can be proved *directly* via the series decomposition of all 24 quantum dilogarithms. The proof is based on several *finite* q-resummations (like q-binomial theorems). This is the first value of the formula (1.37). The second one is that it gives a nice way to derive the finite-dimensional complete R-matrix (simply by replacing ψ and $\mathbf{P}$ by their finite-dimensional counterparts, [21, 22, 24]).

It is important to mention the case of $|q| = 1$. In this case the quantum dilogarithmic functions should be replaced by Faddeev's integral [25]. Briefly, it appears when one considers the Jacoby imaginary transformation of an argument of ψ and q:

$$\mathbf{u} = \mathrm{e}^{\mathrm{i}z} \qquad -q^{1/2} = \mathrm{e}^{\mathrm{i}\pi\theta} \mapsto \tilde{\mathbf{u}} = \mathrm{e}^{\mathrm{i}z/\theta} \qquad -\tilde{q}^{1/2} = \mathrm{e}^{-\mathrm{i}\pi/\theta}. \tag{1.39}$$

Then

$$\psi_F(\mathbf{u}) = \frac{(q^{1/2}\mathbf{u}; q)_\infty}{(\tilde{q}^{1/2}\tilde{\mathbf{u}}; \tilde{q})_\infty} \tag{1.40}$$

and the following expression for $\psi_F(\mathbf{u})$ is valid in the limit of real θ [25]:

$$\psi_F(\mathbf{u})(= s(z)) = \exp \frac{1}{4}\int_\infty^\infty \frac{\mathrm{e}^{z\xi}}{\sinh \pi\xi \sinh \pi\theta\xi} \frac{\mathrm{d}\xi}{\xi} \tag{1.41}$$

where the singularity at $\xi = 0$ is circled from above.

1.4. Hamiltonian structure of $\mathbf{R}$

Returning now to map (1.37), the map $\mathbf{R}$ conserves four independent operators:

$$\mathbf{w}_1 \cdot \mathbf{w}_2 \qquad \mathbf{u}_2 \cdot \mathbf{u}_3 \qquad \mathbf{s} \tag{1.42}$$

and

$$\begin{aligned}\mathbf{H} = \mathbf{w}_1 \cdot \mathbf{u}_3^{-1} - q^{1/2}\mathbf{u}_1 \cdot \mathbf{u}_2^{-1} \cdot \mathbf{w}_2 \cdot \mathbf{w}_3^{-1} - \kappa_1 q^{-1/2}\mathbf{u}_1 \cdot \mathbf{w}_1 \cdot \mathbf{u}_2^{-1} \cdot \mathbf{u}_3^{-1} + \kappa_3 \mathbf{u}_1 \cdot \mathbf{u}_2^{-1} \\ -\kappa_2 q^{-1/2}\mathbf{w}_1 \cdot \mathbf{w}_2 \cdot \mathbf{u}_3^{-1} \cdot \mathbf{w}_3^{-1} + \kappa_2 \mathbf{w}_2 \cdot \mathbf{w}_3^{-1} \\ = (\mathbf{W}^{-1} + \kappa_1\mathbf{U} - q^{1/2}\kappa_3\mathbf{U}\mathbf{W}^{-1}) + \mathbf{s}^{-1}(\mathbf{W} + \kappa_2\mathbf{U}^{-1} - q^{1/2}\kappa_2\mathbf{U}^{-1}\mathbf{W}).\end{aligned} \tag{1.43}$$

Actually $\mathbf{R}$ depends only on two of them, $\mathbf{s}$ and $\mathbf{H}$.

Consider the following product:

$$\sigma = \psi(a\mathbf{w}^{-1}) \cdot \psi(b\mathbf{u}) \cdot \psi(-q^{-1/2}c\mathbf{u}\mathbf{w}) \cdot \psi(a'\mathbf{w}) \cdot \psi(b'\mathbf{u}^{-1}). \tag{1.44}$$

Let

$$\chi = a\mathbf{w}^{-1} + a'\mathbf{w} + b\mathbf{u} + b'\mathbf{u}^{-1} - q^{-1/2}c\mathbf{u}\mathbf{w} - q^{-1/2}ab'\mathbf{u}^{-1}\mathbf{w}^{-1}. \tag{1.45}$$

It is easy to check $\sigma \cdot \chi = \chi \cdot \sigma$. Hence σ as an operator is a function on χ:

$$\sigma = \sigma\left(aa', bb', \frac{c}{a'b} \mid \chi\right). \tag{1.46}$$

I did not find an explicit form of function σ, only a special case of σ when $c = b' = 0$: then

$$\psi(a\mathbf{w}^{-1})\psi(b\mathbf{u})\psi(a'\mathbf{w}) = \psi(a\theta^{-1})\psi(a'\theta) \tag{1.47}$$

where

$$a\theta^{-1} + a'\theta = a\mathbf{w}^{-1} + b\mathbf{u} + a'\mathbf{w}. \tag{1.48}$$

Nevertheless, direct calculations give $\mathbf{R}^2$ in terms of the σ introduced. First, it is convenient to rewrite $\mathbf{R}$:

$$\mathbf{R} = \psi(\mathbf{W}^{-1})\psi(-q^{1/2}\kappa_3\mathbf{U}\mathbf{W}^{-1})\mathbf{P}\left(\sqrt{\frac{\kappa_3}{\kappa_2}}\mathbf{U}, \mathbf{s}^{-1}\mathbf{W}^2\right)\psi\left(-q^{1/2}\frac{\kappa_1\kappa_2}{\kappa_3}\mathbf{U}^{-1}\mathbf{W}\right)^{-1}\psi\left(\frac{\kappa_1}{\kappa_3}\mathbf{W}\right)^{-1}. \tag{1.49}$$

Then

$$\mathbf{R}^2 = \mathcal{N}\cdot\mathcal{D}^{-1} \tag{1.50}$$

where

$$\mathcal{N} = \psi(\mathbf{W}^{-1})\psi(-q^{1/2}\kappa_3\mathbf{U}\mathbf{W}^{-1})\psi(\kappa_1\mathbf{U})\psi(\mathbf{s}^{-1}\mathbf{W})\psi(-q^{1/2}\kappa_2\mathbf{s}^{-1}\mathbf{U}^{-1}\mathbf{W}) \tag{1.51}$$

and

$$\mathcal{D} = \psi\left(\frac{\kappa_1}{\kappa_3}\mathbf{W}\right)\psi\left(-q^{1/2}\frac{\kappa_1\kappa_2}{\kappa_3}\mathbf{U}^{-1}\mathbf{W}\right)\psi\left(\frac{\kappa_1\kappa_2}{\kappa_3}\mathbf{U}^{-1}\right)\psi\left(\frac{\kappa_1}{\kappa_3}\mathbf{s}\mathbf{W}^{-1}\right)\psi(-q^{1/2}\kappa_1\mathbf{s}\mathbf{U}\mathbf{W}^{-1}). \tag{1.52}$$

Comparing these with the definition of σ, we obtain

$$\mathcal{N} = \sigma\left(\mathbf{s}^{-1}, \kappa_2\kappa_3\mathbf{s}^{-1}, \frac{\kappa_1}{\kappa_3}\mathbf{s} \mid \mathbf{H}\right) \qquad \mathcal{D} = \sigma\left(\frac{\kappa_1^2}{\kappa_3^2}\mathbf{s}, \frac{\kappa_1^2\kappa_2}{\kappa_3}\mathbf{s}, \frac{\kappa_3}{\kappa_1}\mathbf{s}^{-1} \mid \frac{\kappa_1}{\kappa_3}\mathbf{s}\mathbf{H}\right) \tag{1.53}$$

where $\mathbf{H}$ is given by (1.43).

2. Evolution system

In this section we apply operator $\mathbf{R}$ defined in the previous section to construct an evolution model explicitly. Due to the current system's background we formulate this model in terms of the regular lattice defined on the torus, its motion, its current system and so on.

The main result of our paper is the generating function for the integrals of motion for the evolution. The derivation of the integrals is based on the auxiliary linear problem.

2.1. Kagomé lattice on the torus

An example of a regular lattice which contains both $\triangle$- and $\bigtriangledown$-type triangles is the so-called Kagomé lattice. As was mentioned in the introduction, the Kagomé lattices appear in the sections of the regular 3D cubic lattices made by inclined planes. Thus the Kagomé lattice and its evolution actually corresponds to the rectangular 3D lattice and thus is quite natural. The Kagomé lattice consists of three sets of parallel lines: the usual situation is shown in figure 4. The sites of the lattice are both $\triangle$ and $\bigtriangledown$ triangles, and hexagons.

For a given lattice introduce the labelling for the vertices; mark the $\triangle$ triangles by the point notation P, and let a and b are the multiplicative shifts in the northern and eastern directions, so that the elementary shift in the south-east direction is $c = a^{-1}b$. Nearest to triangle P are triangles aP, bP, cP, $a^{-1}P$, $b^{-1}P$ and $c^{-1}P$. Some of them are shown in figure 4. The multiplicative notations for the coordinates looks a little strange: we use them simply to make our formulae shorter.

For three vertices surrounding the $\triangle$-type triangle P introduce the notation $(1, P)$, $(2, P)$ and $(3, P)$. We use this notation for everything assigned to the vertices.

We define the Kagomé lattice on the torus of size M; formally this means the following equivalence:

$$a^M P \sim b^M P \sim c^M P \sim P. \tag{2.1}$$

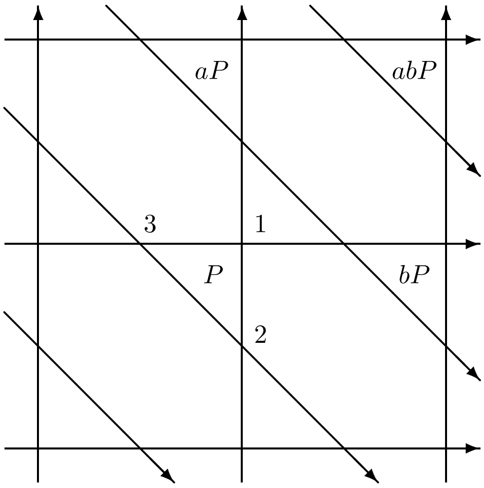

Figure 4. The Kagomé lattice.

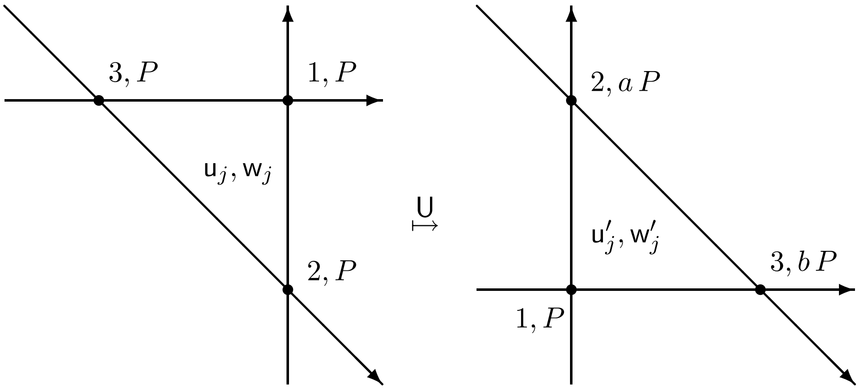

Figure 5. Geometrical representation of evolution.

Given the notion of the equivalence, we may consider the shifts of all inclined lines through the rectangular vertices in the north-eastern direction as shown in figure 5. It is easy to see that figure 5 is equivalent to figure 2. The structure of the Kagomé lattice conserves by such shifts being made simultaneously for all $\triangle$, but the marking of the vertices changes a little. This is visible in figure 5.

We now give a pure algebraic definition of the evolution. The phase space of the system is the set of $3M^2$ Weyl pairs $\mathbf{u}_{j,P}$ and $\mathbf{w}_{j,P}$, $j = 1, 2, 3$, $P = a^\alpha b^\beta P_0$, where P_0 is some frame of the reference's distinguished point, and the toroidal boundary conditions mean

$$\begin{aligned} \mathbf{u}_{j,a^M P} &= \mathbf{u}_{j,b^M P} = \mathbf{u}_{j,P} \\ \mathbf{w}_{j,a^M P} &= \mathbf{w}_{j,b^M P} = \mathbf{w}_{j,P}. \end{aligned} \tag{2.2}$$

The phase space is quantized by the definition. Let $\mathbf{u}'_{j,P}, \mathbf{w}'_{j,P}$ for fixed P be given by (1.16), so that the map $\{\mathbf{u}_{j,P}, \mathbf{w}_{j,P}\} \mapsto \{\mathbf{u}'_{j,P}, \mathbf{w}'_{j,P}\}$ is given by the operator

$$\mathcal{R} = \prod_P \mathbf{R}_P \tag{2.3}$$

where $\mathbf{R}_{P'}$ acts trivially on the variables of any triangle $P \neq P'$. Note, we suppose that $\kappa_{j,P}$

do not depend on P,

$$\kappa_{j,P} = \kappa_j \tag{2.4}$$

so that with respect to κ the translation invariance of the lattice is assumed. We define the action of the evolution operator $\mathbf{U}$ as follows:

$$\begin{cases} \mathbf{U}\cdot\mathbf{u}_{1,P}\cdot\mathbf{U}^{-1} = \mathbf{u}'_{1,P} & \mathbf{U}\cdot\mathbf{w}_{1,P}\cdot\mathbf{U}^{-1} = \mathbf{u}'_{1,P} \\ \mathbf{U}\cdot\mathbf{u}_{2,P}\cdot\mathbf{U}^{-1} = \mathbf{u}'_{2,a^{-1}P} & \mathbf{U}\cdot\mathbf{w}_{2,P}\cdot\mathbf{U}^{-1} = \mathbf{w}'_{2,a^{-1}P} \\ \mathbf{U}\cdot\mathbf{u}_{3,P}\cdot\mathbf{U}^{-1} = \mathbf{u}'_{3,b^{-1}P} & \mathbf{U}\cdot\mathbf{w}_{3,P}\cdot\mathbf{U}^{-1} = \mathbf{w}'_{3,b^{-1}P}. \end{cases} \tag{2.5}$$

This identification means that $\mathbf{U}\cdot\mathbf{u}_{j,P}\cdot\mathbf{U}^{-1}$ and $\mathbf{U}\cdot\mathbf{w}_{j,P}\mathbf{U}^{-1}$ are the variables which appear in place of $\mathbf{u}_{j,P}, \mathbf{w}_{j,P}$ of figure 5. We take the primary variables $\{\mathbf{u}_{j,P}, \mathbf{w}_{j,P}\}$ of the given lattice as the initial data for the discrete time evolution,

$$\mathbf{u}_{j,P} = \mathbf{u}_{j,P}(0) \qquad \mathbf{w}_{j,P} = \mathbf{w}_{j,P}(0). \tag{2.6}$$

The evolution from $t = n$ to $t = n + 1$ is simply

$$\mathbf{u}_{j,P}(n+1) = \mathbf{U}\cdot\mathbf{u}_{j,P}(n)\cdot\mathbf{U}^{-1} \qquad \mathbf{w}_{j,P}(n+1) = \mathbf{U}\cdot\mathbf{w}_{j,P}(n)\cdot\mathbf{U}^{-1}. \tag{2.7}$$

Clearly, the map $\mathbf{U}$ is the canonical map for the Weyl algebrae, so that $\mathbf{U}$ is the quantum evolution operator. Henceforth we mainly consider the situation for $t = 0$ and the map from $t = 0$ to $t = 1$. We omit the time variable and write f instead of $f(0)$ and $f^\star = \mathbf{U}\cdot f\cdot\mathbf{U}^{-1}$ instead of $f(1)$ for any object f. Due to the homogeneity of evolution (2.7), (2.5) our considerations appear to be valid for a situation with $t = n$ and the map from $t = n$ to $t = n + 1$.

2.2. Linear system

We now investigate the linear system for the Kagomé lattice on the torus.

Assign to the vertex (j, P) of the primary $(t = 0)$ Kagomé lattice the internal current $\phi_{j,P}$. The linear system is the set of $3M^2$ linear homogeneous equations for $3M^2$ internal currents

$$\begin{aligned} f_{1,P} &\equiv \mathbf{w}_{1,P}\cdot\phi_{1,P} + \phi_{2,P} + q^{1/2}\mathbf{u}_{3,P}\cdot\phi_{3,P} = 0 \\ f_{2,P} &\equiv q^{1/2}\mathbf{u}_{1,P}\cdot\phi_{1,P} + \kappa_2\mathbf{u}_{2,aP}\mathbf{w}_{2,aP}\cdot\phi_{2,aP} + \mathbf{w}_{3,bP}\cdot\phi_{3,bP} = 0 \\ f_{3,P} &\equiv \phi_{1,a^{-1}P} + \kappa_1\mathbf{u}_{1,b^{-1}P}\mathbf{w}_{1,b^{-1},P}\cdot\phi_{1,b^{-1}P} + \mathbf{w}_{2,P}\cdot\phi_{2,P} + q^{1/2}\mathbf{u}_{b,b^{-1}P}\cdot\phi_{2,b^{-1}P} \\ &\qquad +\phi_{3,a^{-1}P} + \kappa_3\mathbf{u}_{3,P}\mathbf{w}_{3,P}\cdot\phi_{3,P} = 0. \end{aligned} \tag{2.8}$$

Here we have introduced absolutely inessential notation $f_{j,P}$ simply in order to distinguish these equations. $f_{j,P}$ are assigned to the sites. Due to the homogeneity we may impose *the quasiperiodical boundary conditions* for $\phi_{j,P}$:

$$\phi_{j,a^MP} = A\phi_{j,P} \qquad \phi_{j,b^MP} = B\phi_{j,P}. \tag{2.9}$$

It is useful to rewrite this system in matrix form, $F \equiv \boldsymbol{L}\cdot\Phi = 0$. First combine $\phi_{j,P}$ with the same j into the column vector Φ_j with M^2 components, as $(\Phi_j)_P = \phi_{j,P}$. Introduce matrices T_a and T_b as

$$(T_a\cdot\Phi_j)_P = \phi_{j,aP} \qquad (T_b\cdot\Phi_j)_P = \phi_{j,bP}. \tag{2.10}$$

Due to (2.9)

$$T_a^M = A \qquad T_b^M = B. \tag{2.11}$$

Combine further $\mathbf{u}_{j,P}$ and $\mathbf{w}_{j,P}$ with the same j into diagonal matrices $\mathbf{u}_j$ and $\mathbf{w}_j$ with the same ordering of P as in the definition of Φ_j,

$$\mathbf{u}_j = \mathrm{diag}_P\mathbf{u}_{j,P} \qquad \mathbf{w}_j = \mathrm{diag}_P\mathbf{w}_{j,P}. \tag{2.12}$$

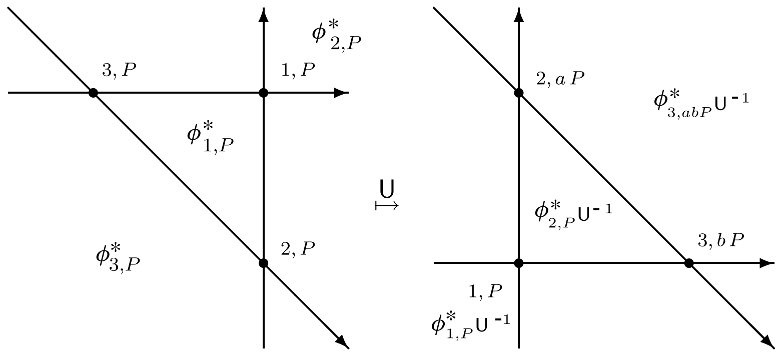

Figure 6. Co-currents on the lattice.

Obviously,

$$(T_a \cdot \mathbf{u}_j \cdot T_a^{-1})_P = \mathbf{u}_{j,aP} \qquad (T_b \cdot \mathbf{u}_j \cdot T_b^{-1})_P = \mathbf{u}_{j,bP} \tag{2.13}$$

and the same for $\mathbf{w}_j$.

Next combine Φ_1, Φ_1, Φ_3 into $3M^2$ column Φ. Then from (2.8) the matrix $\boldsymbol{L}$ can be extracted in the $3 \times 3M^2 \times M^2$ block form:

$$\boldsymbol{L} = \begin{pmatrix} \mathbf{w}_1 & 1 & q^{1/2}\mathbf{u}_3 \\ q^{1/2}\mathbf{u}_1 & T_a\kappa_2\mathbf{u}_2\mathbf{w}_2 & T_b\mathbf{w}_3 \\ T_a^{-1} + T_b^{-1}\kappa_1\mathbf{u}_1\mathbf{w}_1 & \mathbf{w}_2 + T_b^{-1}q^{1/2}\mathbf{u}_2 & T_a^{-1} + \kappa_3\mathbf{u}_3\mathbf{w}_3 \end{pmatrix}. \tag{2.14}$$

Recall, system $\boldsymbol{L} \cdot \Phi = 0$ is $3M^2$ equations for $3M^2$ components of Φ.

Introduce now co-currents. System $\boldsymbol{L} \cdot \Phi = 0$ one may regard as the equations of motion for the 2D system with the action

$$\mathcal{A} \equiv \Phi^* \cdot \boldsymbol{L} \cdot \Phi. \tag{2.15}$$

The block form of the co-currents Φ^* is thus fixed from the form of $\boldsymbol{L}$, or from (2.8). Equations of motion for Φ^* are $F^* \equiv \Phi^* \cdot \boldsymbol{L} = 0$, and in component form

$$\begin{aligned} f^*_{1,P} &\equiv \phi^*_{1,P} \cdot \mathbf{w}_{1,P} + \phi^*_{2,P} \cdot q^{1/2}\mathbf{u}_{1,P} + \phi^*_{3,aP} + \phi^*_{3,bP} \cdot \kappa_1\mathbf{u}_{1,P}\mathbf{w}_{1,P} \\ f^*_{2,P} &\equiv \phi^*_{1,P} + \phi^*_{2,a^{-1}P} \cdot \kappa_2\mathbf{u}_{2,P}\mathbf{w}_{2,P} + \phi^*_{3,P} \cdot \mathbf{w}_{2,P} + \phi^*_{3,bP} \cdot q^{1/2}\mathbf{u}_{2,P} \\ f^*_{3,P} &\equiv \phi^*_{1,P} \cdot q^{1/2}\mathbf{u}_{3,P} + \phi^*_{2,b^{-1}P} \cdot \mathbf{w}_{3,P} + \phi^*_{3,aP} + \phi^*_{3,P} \cdot \kappa_3 u_{3,P}\mathbf{w}_{3,P}. \end{aligned} \tag{2.16}$$

Here $f^*_{j,P}$ corresponds to (j, P)th vertex. The assignment of the co-currents is shown in figure 6.

Elements of $F^* = \Phi^* \cdot \boldsymbol{L}$ have the following remarkable feature: coefficients in $f^*_{j,P}$ belong to the algebra of $\mathbf{u}_{j,P}$, $\mathbf{w}_{j,P}$ only. This means that the elements of any two columns of $\boldsymbol{L}$ commute. Hence the object

$$\mathbf{d}et(\boldsymbol{L}) = \sum_\sigma (-)^\sigma \prod_\alpha \boldsymbol{L}_{\alpha,\sigma(\alpha)} \tag{2.17}$$

where $\alpha \in (j, P)$ and σ are all the permutations, is well defined. One can prove the following proposition.

Proposition 2. *The admissibility condition for the linear homogeneous system* $\Phi^* \cdot \boldsymbol{L} = 0$ *is*

$$\phi^*_{j,P} \cdot \mathbf{d}et(\boldsymbol{L}) = 0 \tag{2.18}$$

for all (j, P).

It is important that, due to $T_a^M = A$ and $T_b^M = B$, $\mathbf{d}et(\boldsymbol{L})$ is a Laurent polynomial with respect to the quasimomenta A and B.

2.3. Evolution of the co-currents and integrals of motion

We now consider the shift of the inclined lines giving the evolution. The internal currents as well as the co-currents change, and we can trace these changes.

Introduce two extra matrices, $\boldsymbol{K}$ and $\boldsymbol{M}$:

$$\boldsymbol{K} = \begin{pmatrix} 0 & \Lambda_0 & 0 \\ 0 & 0 & T_a T_b \\ 1 & K_{3,2} & 0 \end{pmatrix} \tag{2.19}$$

where

$$\Lambda_0 = \frac{\kappa_1}{\kappa_2} q^{-1/2} \mathbf{w}_1 \mathbf{u}_2^{-1} \mathbf{w}_3^{-1} + \frac{\kappa_3}{\kappa_2} \mathbf{u}_1^{-1} \mathbf{w}_2^{-1} \mathbf{u}_3 \tag{2.20}$$

$$K_{3,2} = T_a^{-1} q^{-1/2} \Lambda_2 + \frac{\kappa_3}{\kappa_2} \Lambda_1 + T_b^{-1} \frac{\kappa_1}{\kappa_2} \Lambda_3 \tag{2.21}$$

with Λ_j standing for the diagonal matrices with the entries given by (1.16) correspondingly, and

$$\boldsymbol{M} = \begin{pmatrix} 0 & \mathbf{u}_1^{-1} \mathbf{u}_2' T_a & q^{-1/2} \mathbf{u}_1^{-1} T_b \\ \frac{\kappa_1}{\kappa_2} \mathbf{w}_2^{-1} \mathbf{u}_2^{-1} \mathbf{u}_1' \mathbf{w}_1' & 0 & \frac{\kappa_3}{\kappa_2} \mathbf{w}_2^{-1} \mathbf{u}_2^{-1} \mathbf{u}_3' \mathbf{w}_3' T_b \\ \mathbf{w}_3^{-1} & \mathbf{w}_3^{-1} \mathbf{w}_2' T_a & 0 \end{pmatrix}. \tag{2.22}$$

Apply the evolution operator $\mathbf{U}$ to $\boldsymbol{L}$:

$$\boldsymbol{L}^\star = \begin{pmatrix} \mathbf{w}_1' & 1 & q^{1/2} T_b^{-1} \mathbf{u}_3' T_b \\ q^{1/2} \mathbf{u}_1' & \kappa_2 \mathbf{u}_2' \mathbf{w}_2' T_a & \mathbf{w}_3' T_b \\ T_a^{-1} + T_b^{-1} \kappa_1 \mathbf{u}_1' \mathbf{w}_1' & T_a^{-1} (\mathbf{w}_2' + T_b^{-1} q^{1/2} \mathbf{u}_2') T_a & T_a^{-1} + T_b^{-1} \kappa_3 \mathbf{u}_3' \mathbf{w}_3' T_b \end{pmatrix}. \tag{2.23}$$

Recall our convention to denote $f^\star \equiv \mathbf{U} \cdot f \cdot \mathbf{U}^{-1}$ for any f. The following relation can be verified directly:

$$\boldsymbol{K} \cdot \boldsymbol{L}^\star = \boldsymbol{L} \cdot \boldsymbol{M}. \tag{2.24}$$

$\boldsymbol{M}$ in general is the matrix making $\mathbf{U}\phi_{j,P} = \phi^\star_{j,P} \mapsto \phi_{j,P}$, and $\boldsymbol{K}$ makes $\phi^*_{j,P} \mapsto \phi^*_{j,P}\mathbf{U}^{-1} = \phi^{*\star}_{j,P}$. Also $\boldsymbol{K}$ and $\boldsymbol{M}$ admit

$$\boldsymbol{K} \mapsto \boldsymbol{K} + \boldsymbol{L} \cdot \boldsymbol{N} \qquad \boldsymbol{M} \mapsto \boldsymbol{M} + \boldsymbol{N} \cdot \boldsymbol{L}^\star \tag{2.25}$$

with arbitrary $\boldsymbol{N}$. One can prove the following proposition.

Proposition 3. *Relation $\boldsymbol{K} \cdot \mathbf{det}(\boldsymbol{L}) = \mathbf{det}(\boldsymbol{L}) \cdot \tilde{\boldsymbol{K}}$ holds for all components of $\boldsymbol{K}$.*

Now we may give the heuristic derivation of the conservation of $\mathbf{d}et(\boldsymbol{L})$. Suppose we solve the linear co-system $\Phi^* \cdot \boldsymbol{L} = 0$ at the time $t = 0$. We then apply the evolution operator to this system, thus we have to obtain the solution of the system

$$\Phi^{*\star} \cdot \boldsymbol{L}^\star = (\Phi^* \cdot \boldsymbol{K}) \cdot \boldsymbol{L}^\star = 0. \tag{2.26}$$

With respect to Φ^* this map is a simple linear map, so the admissibility condition for the evaluated homogeneous liner system must conserve. Thus due to propositions 2 and 3 we may conclude that

$$\mathbf{det}(\mathbf{U} \cdot \boldsymbol{L} \cdot \mathbf{U}^{-1}) = \mathbf{det}(\boldsymbol{L}) \cdot D \tag{2.27}$$

with some operator D. One may hope that D is not too complicated, and (2.27) is not trivial.

Careful analysis of $\boldsymbol{K}$ and $\boldsymbol{M}$ shows that this D does not depend on the quasimomenta A and B. In the functional limit $q^{1/2} \mapsto \pm 1$ one may easily calculate the determinants of $\boldsymbol{K}$ and $\boldsymbol{M}$; both are proportional to $A^M B^M$, and this term cancels from the determinants of the left- and right-hand sides of (2.24). This is so in the quantum case also.

Hence D in (2.27) is a ratio of any A, B—monomials from $\mathbf{det}(\boldsymbol{L})$ and $\mathbf{det}(\boldsymbol{L}^{\star})$. Element D can be extracted, say, from the $A^M B^{-M}$ component of $\det(\boldsymbol{L})$:

$$D = \prod_P \mathbf{u}_{1,P}^{-1} \cdot \prod_P \mathbf{u}_{1,P}^{\star}. \tag{2.28}$$

This means that we can introduce a simple operator d:

$$D = d \cdot d^{\star -1}. \tag{2.29}$$

Choose directly

$$d = \prod_P \mathbf{u}_{1,P}^{-1}. \tag{2.30}$$

Thus

$$\mathbf{J} = \mathbf{det}(\boldsymbol{L}) \cdot d \tag{2.31}$$

is the invariant of the evolution, i.e.

$$\mathbf{U} \cdot \mathbf{J} = \mathbf{J} \cdot \mathbf{U}. \tag{2.32}$$

Decompose $\mathbf{J}$ as a series of A and B,

$$\mathbf{J} = \sum_{\alpha,\beta\in\Pi} A^{\alpha} B^{\beta} \mathbf{J}_{\alpha,\beta} \tag{2.33}$$

where α and β are integers and their domain (Newton's polygon) Π is defined by $|\alpha| \leqslant M$, $|\beta| \leqslant M$ and $|\alpha+\beta| \leqslant M$. Quasimomenta A and B are arbitrary $\mathcal{C}$-numbers, and the invariance of $\mathbf{J}$ means the invariance of each $\mathbf{J}_{\alpha,\beta}$. On the other hand, $\mathbf{J}$ is a functional of the dynamical variables of the lattice, i.e.

$$\mathbf{J}_{\alpha,\beta} = \mathbf{J}_{\alpha,\beta}(\{\mathbf{u}_{j,P}, \mathbf{w}_{j,P}\}). \tag{2.34}$$

Clearly, due to the homogeneity of the lattice these functionals are invariant with respect to the lattice translations, and hence the conservation of $\mathbf{J}$ gives

$$\mathbf{J}_{\alpha,\beta}(\{\mathbf{u}_{j,P}, \mathbf{w}_{j,P}\}) = \mathbf{J}_{\alpha,\beta}(\{\mathbf{U}\cdot\mathbf{u}_{j,P}\cdot\mathbf{U}^{-1}, \mathbf{U}\cdot\mathbf{w}_{j,P}\cdot\mathbf{U}^{-1}\}) \tag{2.35}$$

i.e. functionals $\mathbf{J}_{\alpha,\beta}$ give the integrals of motion in usual sense.

d can be absorbed into det,

$$\mathbf{J} = \mathbf{det}(\boldsymbol{L}^{(0)}) \tag{2.36}$$

where

$$\boldsymbol{L}^{(0)} = \begin{pmatrix} q^{1/2}\mathbf{u}_1^{-1}\mathbf{w}_1 & 1 & q^{1/2}\mathbf{u}_3 \\ 1 & T_a\kappa_2\mathbf{u}_2\mathbf{w}_2 & T_b\mathbf{w}_3 \\ T_a^{-1}q^{-1/2}\mathbf{u}_1^{-1} + T_b^{-1}q^{1/2}\kappa_1\mathbf{w}_1 & \mathbf{w}_2 + T_b^{-1}q^{1/2}\mathbf{u}_2 & T_a^{-1} + \kappa_3\mathbf{u}_3\mathbf{w}_3 \end{pmatrix}. \tag{2.37}$$

The total number of $\mathbf{J}_{\alpha,\beta}$ is $3M^2+3M+1$, and there are $3M^2+1$ independent, and of these one can choose only $3M^2$ commutative, so $\mathbf{J}$ gives the complete set of integrals. The existence of $3M^2$ *Abelian* integrals is the hypothesis tested for small M.

All integrals corresponding to the boundary of domain Π, $|\alpha| = M$, $|\beta| = M$, $|\alpha+\beta| = M$, are equivalent to the following $3M$ elements:

$$\begin{aligned} \overline{u}_j &= \prod_\sigma \mathbf{w}_{1,a^\sigma b^j P_0}^{-1}\mathbf{w}_{2,a^\sigma b^j P_0}^{-1} \\ \overline{v}_j &= \prod_\sigma \mathbf{u}_{2,a^{j+\sigma}b^{-\sigma}P_0}\mathbf{u}_{3,a^{j+\sigma}b^{-\sigma}P_0} \\ \overline{w}_j &= \prod_\sigma \mathbf{u}_{1,a^j b^\sigma P_0}\mathbf{w}_{3,a^j b^\sigma P_0}^{-1} \end{aligned} \tag{2.38}$$

where P_0 is some frame of reference's point as previously. Note, $\overline{v}_j$ are not T_a, T_b-invariant, but restoring this invariance in any way (considering the symmetrical polynomials), one obtains the invariants of $\mathbf{U}$. Between $\overline{w}_j, \overline{u}_j, \overline{v}_j$ one may choose $3M - 1$ commutative elements. The inner part of Π gives $3M^2 - 3M + 1$ highly complicated independent integrals, which gives $g = 3M^2 - 3M + 1$ commutative (up to (2.38)) independent elements. Note, g is the formal genus of the curve $\mathbf{J}(A, B) = \text{const}$.

2.4. Walks on the lattice and the integrals of motion

We now give a geometrical interpretation of the integrals of motion. This interpretation follows directly from the analysis of the determinant. Every integral of motion is a sum of monomials associated with walks on the lattice such that all the walks have the same homotopy class with respect to the torus on which the Kagomé lattice is defined.

It is useful to formulate the walks in terms of general *vertex* variables $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ as in figure 1. Recall the shorter notation $W = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ for the dynamical variables' set. Consider matrix $\boldsymbol{L}$ in this general case. Each row in $\boldsymbol{L}$ corresponds to a vertex of the lattice, and each column of $\boldsymbol{L}$ corresponds to a polygon (i.e. to a site) of the lattice. Thus $\mathbf{d}et(\boldsymbol{L})$ consists on the monomials, each of them corresponds (up to a sign) to a product of different $W_{j,P}$ such that:

- for any vertex (j, P) *only one* of $\mathbf{a}_{j,P}$, $\mathbf{b}_{j,P}$, $\mathbf{c}_{j,P}$, $\mathbf{d}_{j,P}$ is taken in this monomial, and
- for any site *only one* of surrounding $\mathbf{a}, \ldots, \mathbf{d}$ is taken in this monomial.

Take the lattice and mark the places of the vertex variables $\mathbf{a}, \ldots, \mathbf{d}$, corresponding to the monomial, by the arrows, ingoing to the corresponding vertices. Thus, for any site and for any vertex we have only one arrow.

In order to get a purely invariant functional, we have to multiply $\mathbf{d}et(\boldsymbol{L})$ by an integrating monomial; in the general case this monomial is $\prod_P \mathbf{b}_{1,P}^{-1}\mathbf{a}_{2,P}^{-1}\mathbf{a}_{3,P}^{-1}$. This choice of the integrating multiplier corresponds to element d given by (2.30). It is easy to see that this monomial has the same structure as described above. But due to the power -1 we may interpret this monomial geometrically as the set of outgoing arrows.

The system of the outgoing arrows is thus fixed, and shown in figure 7 for each $\triangle$-type triangle of the lattice. For the system of outgoing arrows and any system of ingoing arrows the following is valid:

- for any site there exists exactly one outgoing arrow and exactly one ingoing arrow, and they may touch the same vertex, and
- for any vertex there exists exactly one outgoing arrow and exactly one ingoing arrow, and they may belong to the same site.

Hence there is a unique way to connect all the arrows inside each site so that a walk appears.

So, the walks we consider obey the following demands:

- the system of outlets of the walk is fixed and given by figure 7,
- the walk visits any site only once,
- the walk must visit all the sites and
- the walk must visit all the vertices.

For any walk $\mathcal{W}$ let $\sigma(\mathcal{W})$ be the number of the components of the connectedness (i.e. the number simply connected subwalks). Due to the choice of the outlets, figure 7, we have no connected subwalks with zero homotopy class except the simplest ones: one-step subwalks

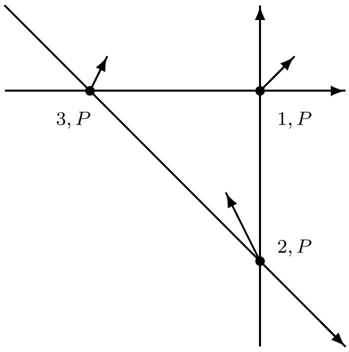

Figure 7. Fixed outlets for the lattice walks.

that leave some vertex into the proper site and return to the same vertex immediately. We call such subwalks trivial loops. They are to be taken into account in the counting of $\sigma(\mathcal{W})$.

Now let walk $\mathcal{W}$ belong to a given homotopy class $\alpha\mathcal{A} + \beta\mathcal{B}$ of the torus, where cycle $\mathcal{A}$ corresponds to T_a^M and cycle $\mathcal{B}$ corresponds to T_b^M, and denote such a walk as $\mathcal{W}_{\alpha,\beta}$.

To a given walk assign a monomial according to the following rules: let the walk pass through vertex (j, P) so that the walk enters the vertex from the side $\mathbf{x} \in W_{j,P}$, and exits the vertex from the side $\mathbf{y} \in W_{j,P}$. Then the multiplier corresponding to (j, P) is $\mathbf{x} \cdot \mathbf{y}^{-1}$. The monomial $\mathbf{J}_{\mathcal{W}}$ is the product of such multipliers corresponding to all the vertices. Thus the reader may see that each monomial we construct gains the structure of an element of $\mathcal{B}'$, described in section 1.1: monomial $\mathbf{J}_{\mathcal{W}}$,

$$\mathbf{J}_{\mathcal{W}} = \ldots \mathbf{x} \cdot \mathbf{y}^{-1} \cdot \mathbf{x}' \cdot \mathbf{y}'^{-1} \ldots . \tag{2.39}$$

$\mathbf{x}$ and $\mathbf{y}$ are assigned to a same vertex, so $\mathbf{x}\cdot\mathbf{y}^{-1}$ does not contain the vertex projective ambiguity, and $\mathbf{y}$ and $\mathbf{x}'$ belong to a same site, so $\mathbf{y}^{-1} \cdot \mathbf{x}'$ does not contain the site ambiguity. Finally, we have to provide the projective invariance of $\mathbf{J}_{\mathcal{W}}$ with respect to the start and end points of each simply connected subwalk. In our case of the local Weyl algebrae this invariance is obvious, because of elements $\mathbf{x} \cdot \mathbf{y}^{-1}$ for different vertices commute.

Trivial loops, obviously, give nothing to $\mathbf{J}_{\mathcal{W}}$, because they correspond to $\mathbf{x} \cdot \mathbf{x}^{-1} = 1$.

With the structure of the walks introduced, the simple analysis of the determinant immediately yields

$$\mathbf{J}_{\alpha,\beta} = \sum_{\mathrm{all}\,\mathcal{W}_{M+\alpha,\beta}} (-)^{\sigma(\mathcal{W}_{M+\alpha,\beta})} \cdot \mathbf{J}_{\mathcal{W}_{M+\alpha,\beta}} \tag{2.40}$$

where the sum is taken over all the walks of the homotopy class $(M + \alpha)\mathcal{A} + \beta\mathcal{B}$ given and the system of the outlets of the walks fixed.

2.5. Example: the Liouville system

As an elementary example of the application of our results consider the following reduced evolution:

- First, consider the evolution system on a strip: in the toroidal conditions $T_a^M = A$ and $T_b^M = B$ the vertical and the horizontal sizes of the torus may not coincide. One may guess $T_a^{M_a} = A$ and $T_b^{M_b} = B$ with different M_a and M_b. Omitting the details concerning least common multipliers etc, note that everything we have done is valid when $M_a = 1$, $M_b = M$.
- Second, we deal with the limit $\kappa_1 \ll \kappa_2 = \kappa_3 \ll 1$; this simplifies all the calculations significantly. Note, a half of the dynamical variables become trivial in this case, and hence a half of the integrals of motion have no interest.

First we write down the action of the evolution operator $\mathbf{U}$ for $\mathbf{R} \mapsto \mathbf{r}$ explicitly. On the strip $T_a \equiv A$, and we use conventional notation for the geometrical coordinate

$$P = b^k P_0 \mapsto k. \tag{2.41}$$

The evolution is given by

$$\begin{aligned}
&\mathbf{U}\cdot\mathbf{w}_{1,k}\cdot\mathbf{U}^{-1} = (1 - q^{1/2}\mathbf{w}_{1,k}^{-1}\mathbf{u}_{3,k})\cdot\mathbf{w}_{1,k}\mathbf{w}_{2,k}\mathbf{w}_{3,k}^{-1}\\
&\mathbf{U}\cdot\mathbf{u}_{1,k}\cdot\mathbf{U}^{-1} = \mathbf{u}_{1,k}\mathbf{w}_{2,k}\mathbf{w}_{3,k}^{-1}\\
&\mathbf{U}\cdot\mathbf{w}_{2,k}\cdot\mathbf{U}^{-1} = \mathbf{w}_{3,k}\cdot(1 - q^{1/2}\mathbf{w}_{1,k}^{-1}\mathbf{u}_{3,k})^{-1}\\
&\mathbf{U}\cdot\mathbf{u}_{2,k}\cdot\mathbf{U}^{-1} = (-q^{1/2}\mathbf{u}_{1,k}\mathbf{w}_{1,k}^{-1}\mathbf{u}_{3,k})\cdot(1 - q^{1/2}\mathbf{w}_{1,k}^{-1}\mathbf{u}_{3,k})^{-1}\\
&\mathbf{U}\cdot\mathbf{w}_{3,k}\cdot\mathbf{U}^{-1} = \mathbf{w}_{2,k-1}\\
&\mathbf{U}\cdot\mathbf{u}_{3,k}\cdot\mathbf{U}^{-1} = (1 - q^{1/2}\mathbf{w}_{1,k-1}^{-1}\mathbf{u}_{3,k-1})\cdot(-q^{1/2}\mathbf{u}_{1,k-1}^{-1}\mathbf{w}_{1,k-1}\mathbf{u}_{2,k-1}).
\end{aligned} \tag{2.42}$$

We now change the variables, introducing the 'observable' ones $\mathbf{a}_k$ and $\mathbf{b}_k$:

$$\mathbf{a}_k = \mathbf{w}_{1,k}^{-1}\cdot\mathbf{u}_{3,k} \qquad \mathbf{b}_k = -q^{1/2}\mathbf{u}_{1,k}^{-1}\cdot\mathbf{u}_{2,k}\cdot\mathbf{w}_{2,k}^{-1}\cdot\mathbf{w}_{3,k+1} \tag{2.43}$$

as well as the centres

$$\overline{u}_k = \mathbf{w}_{1,k}^{-1}\cdot\mathbf{w}_{2,k}^{-1}. \tag{2.44}$$

Nontrivial commutation relations are simply

$$\mathbf{b}_k\cdot\mathbf{a}_k = q\mathbf{a}_k\cdot\mathbf{b}_k \qquad \mathbf{a}_{k+1}\cdot\mathbf{b}_k = q\mathbf{b}_k\cdot\mathbf{a}_{k+1}. \tag{2.45}$$

The centres are the invariants of the evolution, $\mathbf{U}\cdot\overline{u}_k = \overline{u}_k\cdot\mathbf{U}$, and

$$\begin{aligned}
&\mathbf{U}\cdot\mathbf{a}_k\cdot\mathbf{U}^{-1} = \frac{\overline{u}_k}{\overline{u}_{k-1}}(1 - q^{1/2}\mathbf{a}_{k-1})\cdot\mathbf{b}_{k-1}\cdot(1 - q^{1/2}\mathbf{a}_k)^{-1}\\
&\mathbf{U}\cdot\mathbf{b}_k\cdot\mathbf{U}^{-1} = \mathbf{a}_k.
\end{aligned} \tag{2.46}$$

Up to additional parameters $\overline{u}_k$ this map is nothing but the evolution, governed by the quantum Liouville equation 'on the dual lattice in the laboratory frame of the references' according to the terminology of [6, 17].

To clarify this, we draw, as usual, the system $\mathbf{a}_k(t)$, $\mathbf{b}_k(t)$ on a 2D plane so that, for a fixed time, $\mathbf{a}_k$, $\mathbf{b}_k$ are associated with the vertices of the 'horizontal' staircase and the direction of the time $t \mapsto t+1$ corresponds to the elementary translation in the north-west direction, see figure 8.

Obviously, with the time direction chosen, condition $\mathbf{U}\cdot\mathbf{b}_k\cdot\mathbf{U}^{-1} = \mathbf{a}_k$ is trivial. But $\mathbf{U}\cdot\mathbf{a}_k\cdot\mathbf{U}^{-1}$ touches the whole square with three lower vertices $\mathbf{a}_{k-1}$, $\mathbf{b}_k$, $\mathbf{a}_k$, see figure 9, which also shows the four vertices of this square conveniently denoted by the directions of a compass.

With these notations the relation between the vertices of the square can be expressed as

$$\frac{1}{\overline{u}_{NE}}\mathbf{v}_N\cdot(1 - q^{1/2}\mathbf{v}_E) = \frac{1}{\overline{u}_{WS}}(1 - q^{1/2}\mathbf{v}_W)\cdot\mathbf{v}_S. \tag{2.47}$$

This is the well known Liouville relation on the dual lattice up to parameters $\overline{u}$.

In this case a 'good' matrix $\boldsymbol{L}^{(0)}$ of the coefficients of the linear problem (2.37) can be obtained with the simple limit $\kappa_j = 0$ and $T_a = A$. Now the determinant may be calculated combinatorially. The number of the dynamical variables $\mathbf{a}$, $\mathbf{b}$ is $2M$, so as a result we expect the existence of M integrals of motion.

We introduce auxiliary notations: for $k \leqslant m \leqslant k+M$ let

$$\mathbf{F}_{k-1,m+1} = \sum_{\sigma=k}^{m+1}\mathbf{a}_k\ldots\mathbf{a}_{\sigma-1}\cdot\mathbf{b}_\sigma\ldots\mathbf{b}_m - q^{1/2}\sum_{\sigma=k}^{m}\mathbf{a}_k\ldots\mathbf{a}_{\sigma-1}\cdot\mathbf{a}_\sigma\cdot\mathbf{b}_\sigma\cdot\mathbf{b}_{\sigma+1}\ldots\mathbf{b}_m. \tag{2.48}$$

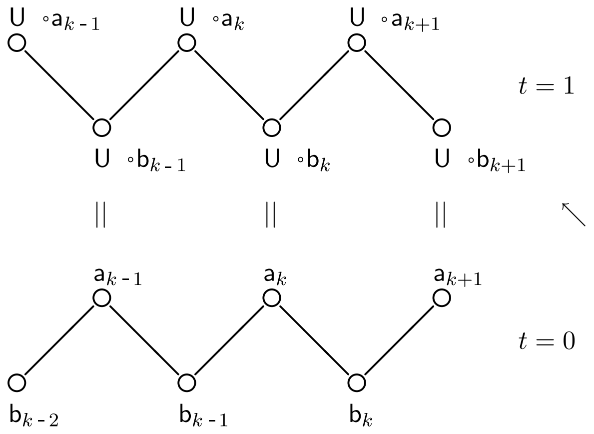

Figure 8. The evolution as the map of the staircase from $t = 0$ to $t = 1$. The 'time direction' is north-west.

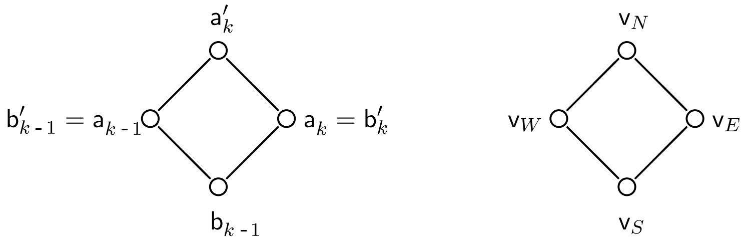

Figure 9. Elementary square, $\mathbf{U} \circ \mathbf{a} \equiv \mathbf{a}'$, $\mathbf{U} \circ \mathbf{b} \equiv \mathbf{b}'$.

Some of $\mathbf{F}_{k-1,m+1}$ for small $m - k$ are given by

$$\mathbf{F}_{k,k+1} = 1 \qquad \mathbf{F}_{k-1,k+1} = \mathbf{a}_k + \mathbf{b}_k - q^{1/2}\mathbf{a}_k \cdot \mathbf{b}_k. \tag{2.49}$$

Further,

$$\mathbf{F}_{k-1,k+2} = \mathbf{a}_k \cdot \mathbf{a}_{k+1} + \mathbf{a}_k \cdot \mathbf{b}_{k+1} + \mathbf{b}_k \cdot \mathbf{b}_{k+1} - q^{1/2}\mathbf{a}_k \cdot \mathbf{a}_{k+1} \cdot \mathbf{b}_{k+1} - q^{1/2}\mathbf{a}_k \cdot \mathbf{b}_k \cdot \mathbf{b}_{k+1} \tag{2.50}$$

and so on. In general, all $\mathbf{F}$ are defined by the recursion relations

$$\mathbf{F}_{k-1,m+1} = \mathbf{a}_k\mathbf{a}_{k+1}\ldots\mathbf{a}_m \cdot (1 - q^{1/2}\mathbf{b}_m) + \mathbf{F}_{k-1,m} \cdot \mathbf{b}_m \tag{2.51}$$

or, equivalently,

$$\mathbf{F}_{k-1,m+1} = (1 - q^{1/2}\mathbf{a}_k) \cdot \mathbf{b}_k\mathbf{b}_{k+1}\ldots\mathbf{b}_m + \mathbf{a}_k \cdot \mathbf{F}_{k,m+1}. \tag{2.52}$$

The integrals, $\mathbf{U} \cdot \mathcal{I}_j = \mathcal{I}_j \cdot \mathbf{U}$, are given by

$$\begin{aligned}
\mathcal{I}_M &= \prod_k \mathbf{a}_k + \prod_k \mathbf{b}_k \\
\mathcal{I}_{M-1} &= \sum_k \overline{u}_k \mathbf{F}_{k,k+M} \\
\mathcal{I}_{M-2} &= \sum_{k<m<k+M} \overline{u}_k \mathbf{F}_{k,m} \overline{u}_m \mathbf{F}_{m,k+M} \\
\mathcal{I}_{M-3} &= \sum_{k<m<n<k+M} \overline{u}_k \mathbf{F}_{k,m} \overline{u}_m \mathbf{F}_{m,n} \overline{u}_n \mathbf{F}_{n,k+M}
\end{aligned} \tag{2.53}$$

and so on, where in all sums the Z_M-cyclicity is implied.

The integrals in the list (2.53) with small indices are remarkably simple, for example

$$\mathcal{I}_1 = \left(\prod_\sigma \overline{u}_\sigma\right) \cdot \left(\sum_k \overline{u}_k^{-1}(\mathbf{a}_k - q^{1/2}\mathbf{a}_k \cdot \mathbf{b}_k + \mathbf{b}_k)\right). \tag{2.54}$$

The conservation of all these integrals may be proven directly.

The combinatorial origin of $\mathcal{I}_1$ is the following: the local contributions to its $\overline{u}_k$th component (in our notations $\mathbf{F}_{k-1,k+1}$) are given by three different diagrams of the homotopy class $\mathcal{A}$ in the graphical representation. These diagrams are simple, because of homotopy class is small and the possible diagrams consist mostly of trivial loops. We call these three nontrivial diagrams 'a particle'. The next integral, $\mathcal{I}_{M-2}$, has the general position —a host of cases when the particles are situated in remote places on the thin torus (the rest are trivial loops). The corresponding contribution to $\mathcal{I}_{M-2}$ is $\overline{u}_k^{-1}\mathbf{F}_{k-1,k+1} \cdot \overline{u}_m^{-1}\mathbf{F}_{m-1,m+1}$, $m \gg k$. In the cases when two particles become closed, the counting of the nontrivial diagrams changes, so that they form a two-particle cluster $\mathbf{F}_{k-1,k+2}$, formed by five nontrivial diagrams only (instead of nine diagrams in the general case). Finally, we consider an n-particle cluster, for which the recursion relations may be easily derived. Thus, we obtain the complete set of integrals in terms of n-particle clusters (2.53).

3. Discussion

We conclude this paper with an overview of the problems to be solved and the aims to be reached. The approach proposed suggests a way to their solution.

First, we mention the problems of the classification of the map

$$\mathbf{R} : \{\mathbf{a}_j, \mathbf{b}_j, \mathbf{c}_j, \mathbf{d}_j\} \mapsto \{\mathbf{a}'_j, \mathbf{b}'_j, \mathbf{c}'_j, \mathbf{d}'_j\} \qquad j = 1, 2, 3 \tag{3.1}$$

in general. The aim is to classify all conserving symplectic structures of the body $\mathcal{B}$. We have discussed only the local case, when the variables, assigned to different vertices, commute and the scalars (spectral parameters) are conserved. We suspect that such a case is not unique, and that there could be other ways to remove the projective ambiguity. The simplest case to be investigated would be to consider all the variables $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ for each vertex as matrices with, for example, non-commutative entries, but with these entries commutative for any two vertices. The matrix structure may be common for all vertices, and thus we would have no commutation between different vertices in general. Another simple possibility is another kind of locality: the case when the dynamical variables commute but do not belong to a same site. This would correspond to the duality between the vertex and the site projective invariance. Note, once our locality is imposed, the Weyl structure appears immediately. Thus the Weyl algebra is the consequence of the locality technically, but the principal origin of the Weyl algebra is mysterious.

A purely technical problem to be mentioned is the investigation of the q-hypergeometrical function σ, equations (1.44), (1.46). More generally, the main future aim with regard to evolution models as the calculation of the S-matrix, $S = \mathbf{U}^{2M}$, as well defined algebraical functions of its integrals.

The main problems for immediate investigation are connected with the integrals of $\mathbf{U}$. $\mathbf{J}(A, B)$ does not seem to be constructive. The aim is at least to calculate the spectrum of $\mathbf{J}$. The combinatorial approach does not appear fruitful for the general $M \times M$ torus. A possible approach would be via functional equations for the integrals of motion, which should follow from the determinant or topological representation of $\mathbf{J}$. Another possibility is that a way resembling the Bethe ansatz in 2D might exist in 3D, i.e. a way of a triangulation of $\mathbf{U}$ with a help of some artificial operators. If such a way exists, it must be based on the linear problem derived.

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References

[1] Zamolodchikov A B 1981 Tetrahedron equations and the relativistic *S* matrix of straight strings in 2+1 dimensions *Commun. Math. Phys.* **79**, 489–505

[2] Bazhanov V V and Stroganov Yu G 1982 *D*–simplex equation *Teor. Mat. Fiz.* **52** 105–13

[3] Baxter R J 1986 The Yang–Baxter equations and the Zamolodchikov model *Physica* D **18** 321–47

[4] Bazhanov V V and Baxter R J 1992 New solvable lattice models in three dimensions *J. Stat. Phys.* **69** 453–85

[5] Sergeev S M, Mangazeev V V and Stroganov Yu G 1996 The vertex formulation of the Bazhanov–Baxter model *J. Stat. Phys.* **82**

[6] Faddeev L D and Volkov A Yu 1997 Algebraic quantization of integrable models in discrete space–time *Preprint* hep-th/9710039

[7] Kashaev R M and Reshetikhin N Yu 1997 Affine Toda field theory as 2 + 1 dimansional integrable system *Commun. Math. Phys.* **188** 251–66

[8] Korepanov I G 1995 Algebraic integrable dynamical systems, 2 + 1-dimensional models in wholly discrete space–time, and inhomogeneous models in 2-dimensional statistical physics *Preprint* solv-int/9506003

[9] Korepanov I G 1997 'Some eigenstates for a model associated with solutions of tetrahedron equation I-V *Preprint* solv-int/9701016, 9702004, 9703010, 9704013, 9705005

[10] Korepanov I G 1997 Particles and strings in a 2 + 1-D integrable quantum model *Preprint* solv-int/9712006 *Commun. Math. Phys.* submitted

[11] Baxter R J 1983 On Zamolodchikov's solution of the Tetrahedron equations *Commun. Math. Phys.* **88** 185–205

[12] Mangazeev V V, Kashaev R M and Stroganov Yu G 1993 Star–square and Tetrahedron equations in the Baxter–Bazhanov model *Int. J. Mod. Phys.* A **8** 1399–409

[13] Korepanov I G 1993 Tetrahedral Zamolodchikov algebras corresponding to Baxter's *L*-operators *Commun. Math. Phys.* **154** 85–97

[14] Maillet J-M and Nijhoff F W 1990 Multidimensional lattice integrability and the simplex equations *Proc. Como Conf. on Nonlinear Evolution Equations: Integrability and Spectral Methods* ed A Degasperis, A P Fordy and M Lakshmanan (Manchester: Manchester University Press) pp 537–48

[15] Maillet J-M and Nijhoff F W 1989 Integrability for multidimensional lattice models *Phys. Lett.* B **224** 389–96

[16] Maillet J-M 1990 Integrable systems and gauge theories *Nucl. Phys. (Proc. Suppl.)* B **18** 212–41

[17] Faddeev L and Volkov A Yu 1994 Hirota equation as an example of an integrable symplectic map *Lett. Math. Phys.* **32** 125–35

[18] Sergeev S M 1998 *Lett. Math. Phys.* **45** 113–19
(Sergeev S M 1997 Solutions of the functional tetrahedron equation connected with the local Yang–Baxter equation for the ferro-electric *Preprint* solv-int/9709006)

[19] Sergeev S M 1997 On a two dimensional system associated with the complex of the solutions of the Tetrahedron equation *Preprint* solv-int/9709013 *Int. J. Mod. Phys.* A to appear

[20] Sergeev S M 1999 *Phys. Lett.* A **253** 145–50
(Sergeev S M 1998 3D symplectic map *Preprint* solv-int/9802014)

[21] Sergeev S M 1996 Operator solutions of simplex equations *Proc. X Int. Conf. Problems of Quantum Field Theory (Alushta, 1996)* (Moscow: Joint Institute for Nuclear Research) pp 154–7

[22] Sergeev S M Maillard J-M 1997 Three dimensional integrable models based on modified Tetrahedron equations and quantum dilogarithm *Phys. Lett.* B **405** 55–63

[23] Faddeev L D and Kashaev R M 1994 Quantum dilogarithm *Mod. Phys. Lett.* A **9**

[24] Bazhanov V V and Reshetikhin N Yu 1995 Remarks on the quantum dilogarithm *J. Phys. A: Math. Gen.* **28** 2217

[25] Faddeev L D 1995 Discrete Heisenberg–Weyl group and modular group *Lett. Math. Phys.* **34** 249–54

[26] Bazhanov V V, Sergeev S M and Mangazeev V V 1995 Quantum dilogarithm and the Tetrahedron equation *Preprint* IHEP 95-141

[27] Sergeev S M 1997 Two–dimensional *R*-matrices–descendants of three dimensional *R*-matrices *Mod. Phys. Lett.* A **12** 1393–410

[28] Korepanov I G, Kashaev R M and Sergeev S M 1998 Functional tetrahedron equation *Preprint* solv-int/9801015 TMF to appear

[29] Kashaev R M and Sergeev S M 1998 On pentagon, ten-term, and tetrahedron relations *Commun. Math. Phys.* **195** 309–19

[30] Sergeev S M 1998 *Preprint* solv-int/9811003